# Group Theory

## Week #5, Lecture #18

GROUP ACTIONS ON SETS (continued)

## I Review + Example

**Setup:** · Group $G$ acting on a set $S$ $\qquad G \circlearrowright S$

$$G \times S \xrightarrow{\mu} S$$
$$(g, x) \longrightarrow gx$$

(1) $e \cdot x = x \quad , \forall x \in S$

(2) $g(hx) = (gh)x$ $\quad \forall x \in S, \forall g, h \in G$

Equivalently, a group action is determined by (and determines) the associated permutation representation

$$G \xrightarrow{\varphi} \mathrm{Sym}(S)$$
$$g \longrightarrow (x \mapsto gx)$$

$\varphi$ a homomorphism

The action (or the representation) is faithful if $\varphi$ is injective that is; for all $g \in G$:

$$(gx = x \ , \forall x \in S) \Rightarrow g = e$$

**Example** Let $G$ act on $S = G$ by conjugation : $g * x = gxg^{-1}$ $\quad (g \in G, \ x \in G)$

then perm. rep. is :

$$\varphi : G \longrightarrow \mathrm{Sym}(G) \quad , \quad \varphi(g) = z_g \qquad (\text{where } z_g(x) = gxg^{-1})$$

so this can be viewed as a homomorphism $\bar{\varphi}$, followed by inclusions:

$$\begin{array}{l} G \xrightarrow{\bar{\varphi}} \mathrm{Inn}(G) \\ \quad \varphi \searrow \quad \cap \\ \qquad\quad \mathrm{Aut}(G) \\ \qquad\qquad \cap \\ \qquad\quad \mathrm{Sym}(G) \end{array}$$

$\bar{\varphi}(g) = z_g$

Question Is $\varphi$ faithful; i.e., is $\ker(\varphi) = \ker(\bar{\varphi})$ ?

Answer: not in general, since $\qquad \left(\begin{array}{l} \text{since } \varphi = z \circ \bar{\varphi} \\ \qquad\qquad \text{injective} \\ \varphi \text{ inj} \iff \bar{\varphi} \text{ inj} \end{array}\right)$ exercise

$$\boxed{\ker(\bar{\varphi}) = Z(G)}$$

$$\{g \in G \mid gxg^{-1} = x \ , \forall x \in G\}$$

More precisely: $\qquad$ (Conj. action of $G$ on $G$ is faithful) $\iff Z(G) = \{e\}$

Eg: · $G$ abelian $\Rightarrow Z(G) = G \Rightarrow$ not faithful

($G$ nontrivial) · $G$ simple, non-abelian $\Rightarrow Z(G) = \{e\} \Rightarrow$ faithful

## II Orbits & Stabilizers

**Def** For a group action of $G$ on a set $S$, define:

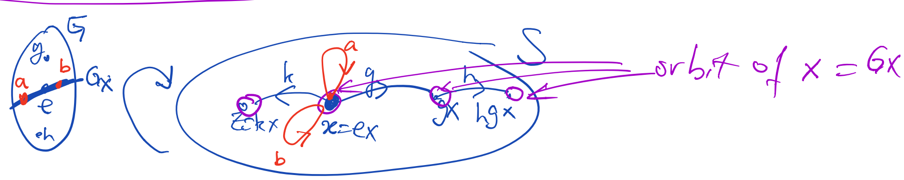
(i) The <u>orbit</u> of an element $x \in S$:

$$\boxed{G \cdot x := \{ s \in S \ : \ s = g \cdot x \ , \text{for some } g \in G \}}$$

(ii) The <u>stabilizer</u> of $x \in S$:

$$\boxed{G_x := \{ g \in G \ : \ gx = x \}}$$

Schematic :




## Examples

(1) $G$ acting on the set $S$ of subgroups of $G$ by conjugation

$$S = \{ H \ : \ H \leq G \} \qquad g * H := gHg^{-1}$$

If $H \trianglelefteq G$ is a normal subgroup, i.e., $gHg^{-1} = H \quad \forall g \in G$

then:

stabilizer: $G_H = \{ g \in G : g * H = H \} = \{ g : gHg^{-1} = H \} = G$

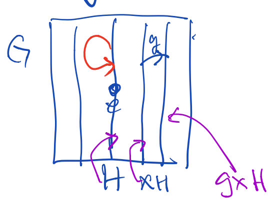
orbit : $G \cdot H = \{ K \leq G : K = g * H \}$

$$= \{ K \leq G \ : \ K = \underset{H}{\underset{\parallel}{gHg^{-1}}} \} = \{ H \}$$

(2) $G$ acting / on the left on the left cosets of a subgroup $H \leq G$

$$S = \{ xH \ : \ x \in G \} \quad , \text{ where } \quad xH = yH \Leftrightarrow y^{-1}x \in H$$

For $g \in G$: $\qquad g * (xH) = (gx)H$




orbit: $G \cdot (xH) = \{ \text{all cosets of } H \} = S$

$\qquad (yx^{-1}) * (xH) = yH \qquad \forall y \in G$

stabilizer: $G_{xH} = \{ g \in G : gxH = xH \}$

$$= \{g : x^{-1} g x H = H\}$$
$$= \{g : x^{-1} g x \in H\} = \{g \mid g \in x H x^{-1}\}$$
$$= x H x^{-1}$$

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**Lemma** The stabilizer of any element $x \in S$ is a subgroup of $G$.

**Proof** Let $G_x = \{g \in G : g * x = x\}$

Let $g, h \in G_x$, i.e.,
$$g * x = x \quad (1)$$
$$h * x = x \iff x = h^{-1} * x \quad (2)$$
(since $h^{-1} * (h * x) = (h^{-1} \cdot h) * x = e * x = x$)

Then: $$(gh^{-1}) * x \underset{\substack{\uparrow \\ \text{axiom (2) for} \\ \text{group actions}}}{=} g * (h^{-1} * x) \underset{(2)}{=} g * x \underset{(1)}{=} x$$

$\therefore \quad G_x \leq H$ **QED**

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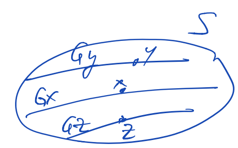
**Lemma** The orbits of a $G$-action on a set $S$ partition $S$:
$$S = \coprod Gx$$




**Proof** Define an equivalence relation on $S$ by:
$$\boxed{x \sim y \overset{\text{def}}{\iff} \exists g \in G \text{ s.t. } g * x = y}$$

Check $\sim$ is an equiv relation:

(i) $x \sim x$ : $e * x = x$ ← (axiom (1))

(ii) $x \sim y \Rightarrow y \sim x$ : $g * x = y \Rightarrow g^{-1} * y = x$ ← (follows from (1) & (2))

(iii) $x \sim y$ & $y \sim z \Rightarrow x \sim z$ : (by (2))

$$g * x = y \ \& \ h * y = z \implies z = h * y = h * (g * x) = (hg) * x$$

✓

The equivalence classes of $\sim$ partition $S$ (into disjoint subsets). So we are left with showing that those equiv. classes are precisely the orbits of $G$ action.

(***) Indeed: $x \sim y \iff y = gx$ for some $g \in G$

$\iff y \in G\cdot x$ (the $G$-orbit of $x$)

$\boxed{\text{QED}}$

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<u>Example</u> $G = \mathbb{C}^* = (\mathbb{C} \setminus \{0\}, \cdot, 1)$ acts on $\mathbb{C}$ by

$$z * x = zx$$

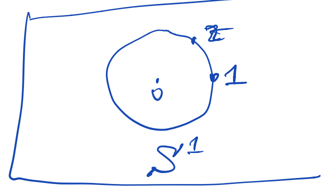
$\left( \begin{array}{l} 1\cdot x = x \ ✓ \\ z\cdot(wx) = (zw)x \ ✓ \end{array} \right)$

By restricting to the unit circle,

$$S^1 = \{ z \in \mathbb{C}^* : |z| = 1 \}$$

$$= \{ e^{i\theta} : 0 \le \theta < 2\pi \}$$

with $e^{i\theta} \cdot e^{i\psi} = e^{i(\theta+\psi)}$




we get an $S^1$-action on $\mathbb{C}$, given by

$z * x = z \cdot x$ (in complex coords)

$e^{i\theta} * re^{i\psi} = re^{i(\theta+\psi)}$ (in polar coords)

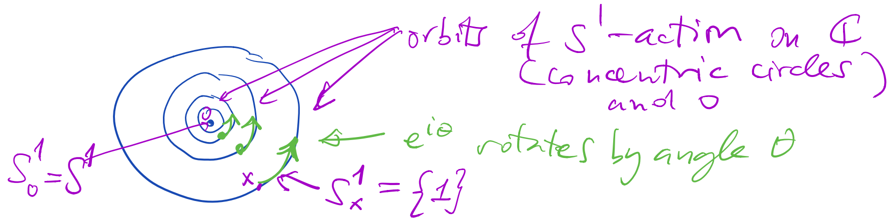
<u>Orbits</u> $S^1x = \{ y \in \mathbb{C} : y = zx, \text{ for some } z \in S^1 \}$

$r \neq 0$: $S^1(re^{i\psi}) = \{ y : y = e^{i\theta} \cdot re^{i\psi} = re^{i(\theta+\psi)} \text{ for some } 0 \le \theta < 2\pi \}$

$= \{ re^{i(\theta+\psi)} : 0 \le \theta < 2\pi \}$

$= \{ \text{circle centered at } 0, \text{ of radius } r \}$

$r = 0$: $S^1 \cdot 0 = \{ y : y = 0\cdot x = 0 \} = \{0\}$




<u>Example</u> $G$ acting on $G$ by conjugation: $g * x = gxg^{-1}$

orbits: $Gx = \{ gxg^{-1} : g \in G \}$ — conjugacy class of $x$

stabilizer: $G_x = \{ g \in G : gxg^{-1} = x \} = C(x)$ — centralizer of $x$

Example $G = GL_n(\mathbb{R})$ acts on $\mathbb{R}^n$ via matrix mult:

$$A * v = A \cdot v$$

- $A$ $n \times n$ matrix with $\det(A) \neq 0$
- $v \in \mathbb{R}^n$ vector
- $A \cdot v$ usual matrix mult

$v = 0$: $G \cdot 0 = \{0\}$

$G_0 = G$

$v \neq 0$ $Gv = \{ w \in \mathbb{R}^n : w = Av$ for some $A \in G \}$

$G_v = \{ A \in G : Av = \overset{\uparrow}{v} \}$

$\lambda=1$ is e-value for $A$

$v$ is eigenvector for $\lambda = 1$